REFLECTION ON THE MAGIC OF e^{Xt}

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In general, given variables X and t, if we raise e to the power Xt, differentiate k times with respect to t, then set $t = 0$, we get back X^k, namely,

$$\frac{d^k}{dt^k} e^{Xt} \bigg|_{t=0} = X^k.$$

This is a consequence of the fact that the Taylor series of e^x is

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \qquad \Longrightarrow \qquad e^{Xt} = \sum_{i=0}^{\infty} \frac{(Xt)^i}{i!}.$$

Mathematicians are quick to realize that if we just raise e to the power Xt, we effectively have stored all the X^k for $k = 0, 1, 2, \ldots$ in one compact expression, e^{Xt}, for later retrieval via differentiation.

WHEN X IS A RANDOM VARIABLE

When X is a random variable, thanks to the *linearity of expectation*, we can store all the *moments*, namely, $\mathbf{E}[X^k]$ for $k = 0, 1, 2, \ldots$ in the expectation of that compact expression, i.e. $\mathbf{E}[e^{Xt}]$. This, of course, is known as the *Moment Generating Function* (MGF) of X with the usual notation:

$$M_X(t) = \mathbf{E}\left[e^{Xt}\right].$$

It is a **moment storage device** — it doesn't generate anything, but one can pull out the kth moment by differentiating k times (and setting $t = 0$), namely,

$$\frac{d^k}{dt^k} M_X(t) \bigg|_{t=0} = \frac{d^k}{dt^k} \mathbf{E}\left[e^{Xt}\right] \bigg|_{t=0} = \mathbf{E}[X^k].$$

As one can see, this works by construction.

MGF IS MORE THAN A MOMENT STORAGE DEVICE

The MGF is far more than a storage for moments; it allows the symbolic composition of distributions, bypassing tedious integration or summation typically required by standard techniques.

Example. A classic demonstration is showing why the sum of independent Normals remains Normal — a process that gets cumbersome when approached via classical convolution integrals.

The MGF of a Normal random variable $X \sim N(\mu, \sigma^2)$ is $e^{\mu t} e^{\sigma^2 t^2/2}$. Let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ be independent. It follows that

$$\begin{aligned}
M_{X+Y}(t) &= \mathbf{E}\left[e^{(X+Y)t}\right] = \mathbf{E}\left[e^{Xt} e^{Yt}\right] \\
&= \mathbf{E}\left[e^{Xt}\right] \mathbf{E}\left[e^{Yt}\right] \qquad \text{(independence)} \\
&= e^{\mu_X t} e^{\sigma_X^2 t^2/2} e^{\mu_Y t} e^{\sigma_Y^2 t^2/2} \\
&= e^{(\mu_X + \mu_Y)t} e^{(\sigma_X^2 + \sigma_Y^2)t^2/2}.
\end{aligned}$$

By the uniqueness of the MGF, this shows $X + Y$ is normally distributed with mean $\mu_X + \mu_Y$ and variance $\sigma_X^2 + \sigma_Y^2$ as desired.

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